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LEAST SQUARES ESTIMATION OF CONDITIONALLY
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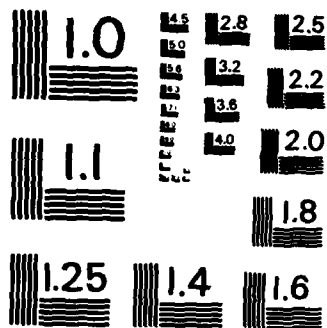
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OF CONDITIONALLY HETEROSCEDASTIC AUTOREGRESSIONS

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OF CONDITIONALLY HETEROSCEDASTIC AUTOREGRESSIONS

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Least Squares Estimation of Conditionally Heteroscedastic
Autoregressions

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Summary

The least squares estimate of the autoregressive parameter of a conditionally heteroscedastic autoregression is consistent and asymptotically normal. Failure to recognize conditional heteroscedasticity results in the underestimation of the variance of the least squares estimate, and in extreme cases, this effect can be substantial. The least squares estimate is not asymptotically "distribution-free", rather, the asymptotic distribution depends on the form of the conditional heteroscedasticity.

Keywords: LEAST SQUARES; AUTOREGRESSION; CONDITIONAL
HETEROSCEDASTICITY

1. Introduction

The classical model for the p -th order autoregression assumes that the innovations that drive the process are independent and identically distributed (i.i.d.). If the innovations have a finite variance, the least squares estimate of the autoregressive parameter, ϕ , is asymptotically unbiased and has an asymptotic normal distribution with a covariance

matrix that depends on Φ alone, and is the same for all distributions of the innovations (Whittle, 1953). Although the least squares estimate is, in this sense, asymptotically "distribution-free", it can be inefficient when the innovations have a heavy-tailed or outlier generating distribution, such as contaminated normal (Martin, 1982). More serious problems arise when "Type I" outliers (Fox, 1972), which affect a single observation, are present. Type I or "additive" outliers can cause the least squares estimate to be seriously biased and can cause the variance to be inflated relative to "robust" estimates (Denby and Martin, 1979).

Outliers are not the only type of model departure that should be of concern. A nonconstant conditional variance can also result in the poor performance of the ordinary least squares estimate. In this paper, the effects of conditional heteroscedasticity are assessed. The properties of conditionally heteroscedastic autoregressions are briefly reviewed in Section 2; a more detailed discussion is given in Nemec (1984). The asymptotic distribution of the least squares estimate is derived for the general conditionally heteroscedastic autoregression, in Section 3, and for two specific examples in Section 4. The asymptotic properties are compared to the small sample properties, as inferred from a limited Monte Carlo simulation in Section 4.1.

2. Conditionally Heteroscedastic Autoregressions

Consider the p -th order zero mean autoregression:

$$y_t = \phi_1 y_{t-1} + \dots + \phi_p y_{t-p} + \varepsilon_t. \quad (2.1)$$

In the classical model, the innovations $\{\varepsilon_t\}$ are assumed to be i.i.d. $N(0, \sigma_\varepsilon^2)$ random variables where $0 < \sigma_\varepsilon^2 < \infty$. If

$\mathcal{V}_t = \sigma(y_t, y_{t-1}, \dots)$ is the sigma-algebra generated by y_t, y_{t-1}, \dots then, given \mathcal{V}_{t-1} , y_t is conditionally $N(\phi_1 y_{t-1} + \dots + \phi_p y_{t-p}, \sigma_\varepsilon^2)$.

The conditional mean is linear in the past observations while the conditional variance is implicitly assumed to be constant.

In certain situations it may be unrealistic to assume that the conditional variance is independent of the past. This paper examines the behaviour of the least squares estimate of $\Phi = (\phi_1, \dots, \phi_p)^T$ when the constant conditional variance assumption is relaxed.

The autoregression (2.1) is said to be conditionally heteroscedastic if, given \mathcal{V}_{t-1} , the conditional variance of y_t is a nonconstant function of the past observations. The resulting conditionally heteroscedastic autoregression or CHAR process is defined by (2.1) where

$$\varepsilon_t | \mathcal{V}_{t-1} \sim N(0, h_{t-1}) \quad (2.2)$$

and

$$h_{t-1} = h(y_{t-1}, y_{t-2}, \dots, y_{t-k}; \theta) \quad 1 \leq k \leq \infty. \quad (2.3)$$

The function h is some positive function of a finite number of the past observations and θ is a vector of parameters.

The process defined by (2.1), (2.2) and (2.3) is not necessarily stationary. To date, conditions that guarantee stationarity have been derived only in special cases (Nemec, 1984) and it is difficult to see how these proofs might be generalized to include an arbitrary function h . Hence it will be necessary to determine if the process is stationary for each h under consideration.

Regardless of whether or not the process is stationary, the innovations of a CHAR process are uncorrelated, although they are obviously not independent. Consequently, the covariance structure of a CHAR process is the same as that of the usual autoregression. In particular, if the process is stationary with a finite variance, the autocorrelation function is

$$\rho_i = \text{Corr}(y_t, y_{t-i}) = A_1 G_1^i + \dots + A_p G_p^i \quad (2.4)$$

where A_1, A_2, \dots, A_p are constants and $G_1^{-1}, G_2^{-1}, \dots, G_p^{-1}$ are the roots of the characteristic polynomial

$$\phi(B) = 1 - \phi_1 B - \dots - \phi_p B^p. \quad (2.5)$$

The stationary variance is

$$\sigma_y^2 = \text{Var}(y_t) = E(h_{t-1}) / (1 - \phi_1 \rho_1 - \dots - \phi_p \rho_p), \quad (2.6)$$

which is finite if $E(h_{t-1})$ is finite.

In the remainder of this paper attention will be restricted to two examples of the CHAR process. Both are stationary processes if appropriate restrictions are placed on the parameters. The properties of the two processes, denoted CHARI and CHARII, are summarized in the following two sections. A more extensive discussion can be found in Nemec (1984).

2.1 The CHARI Process

A CHARI(p,k) process is defined by

$$\text{CHARI}(p,k): y_t = \phi_1 y_{t-1} + \dots + \phi_p y_{t-p} + \varepsilon_t \quad (2.7a)$$

$$\text{where } \varepsilon_t | \mathcal{V}_{t-1} \sim N(0, \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \dots + \alpha_k \varepsilon_{t-k}^2), \quad 1 \leq k < \infty. \quad (2.7b)$$

In this model the conditional variance depends on a finite number of the past innovations in a way that is intuitively reasonable and was first proposed by Engle (1982). If $\alpha_0 > 0$; $\alpha_i \geq 0$, $1 \leq i \leq k$, the ε_t process in (2.7b) is well-defined without reference to the autoregression $\{y_t\}$ and has been called an "autoregressive-conditional-heteroscedastic" (ARCH) process (Engle, 1982). Engle (1982) employed ARCH errors in a general regression model, which includes the CHARI process as a special case.

The stationarity and ergodicity of a CHARI process is given in the following theorem.

Theorem 2.1 (Nemec, 1984) - Let $\{y_t\}_{t=0}^{\infty}$ be a CHARI(p,k) process with $\alpha_0 > 0$ and $\alpha_i \geq 0$, $1 \leq i \leq k$. Assume that all the roots of both polynomials $\Phi(B)$ and $A(B)$ lie outside the unit circle and, y_0, y_1, \dots, y_{p-1} and $\epsilon_0, \epsilon_1, \dots, \epsilon_{k-1}$ have finite variances, where $\Phi(B)$ was defined in (2.5) and $A(B) = 1 - \alpha_1 B - \dots - \alpha_k B^k$. Then $\{y_t\}$ is asymptotically stationary and ergodic.

Under the assumptions of Theorem 2.1, the innovations process $\{\epsilon_t\}$ is stationary and has a finite variance given by

$$\sigma_{\epsilon}^2 = \text{Var}(\epsilon_t) = \alpha_0 / (1 - \alpha_1 - \dots - \alpha_k), \quad (2.8)$$

where $(\alpha_1 + \dots + \alpha_k) < 1$. Substituting this expression into (2.6) gives

$$\sigma_y^2 = \alpha_0 / \{(1 - \alpha_1 - \dots - \alpha_k)(1 - \phi_1 \rho_1 - \dots - \phi_p \rho_p)\}, \quad (2.9)$$

where $(\phi_1 \rho_1 + \dots + \phi_p \rho_p) < 1$.

2.2 The CHARII Process

The second CHAR process is defined by

$$\text{CHARII}(p): y_t = \phi_1 y_{t-1} + \dots + \phi_p y_{t-p} + \epsilon_t \quad (2.10a)$$

$$\text{where } \epsilon_t | \mathcal{V}_{t-1} \sim N\{0, \beta_0 + \beta_1 (\phi_1 y_{t-1} + \dots + \phi_p y_{t-p})^2\}. \quad (2.10b)$$

The parameters β_0 and β_1 are nonnegative with β_0 strictly positive to ensure that the conditional variance is positive with probability one. In a CHARII(p) model the conditional

variance is a simple function of the conditional mean. The same dependence of variance on mean has been used to model heteroscedasticity in the ordinary regression model (see Carroll and Ruppert, 1982, for example).

A CHARII(p) process is stationary and ergodic if the hypotheses of the following theorem hold.

Theorem 2.2 (Nemec, 1984) - Let $\{y_t\}_{t=0}^{\infty}$ be a CHARII(p) process with $\beta_0 > 0$ and $\beta_1 \geq 0$. Assume that all the roots of $\Phi(B)$ lie outside the unit circle and y_0, y_1, \dots, y_{p-1} have finite variances. Then $\{y_t\}$ is asymptotically stationary and ergodic if $(1+\beta_1)(\phi_1\rho_1 + \dots + \phi_p\rho_p) < 1$ where $\rho_i, 1 \leq i \leq p$, are given by (2.4).

The variance of a stationary CHARII(p) process is finite and is given by

$$\sigma_y^2 = \beta_0 / \{1 - (1+\beta_1)(\phi_1\rho_1 + \dots + \phi_p\rho_p)\} \quad (2.11)$$

where $(1+\beta_1)(\phi_1\rho_1 + \dots + \phi_p\rho_p) < 1$.

The CHARI and CHARII processes are two examples of the general class of conditionally heteroscedastic autoregressions. These particular examples were chosen because they are both plausible and mathematically convenient.

3. Least Squares Estimation

Let $\{y_t\}$ be the autoregression in (2.1) where the order (p) is assumed to be known. A least squares estimate $\hat{\Phi}_{LS}^{(n)}$ of the autoregressive parameter, based on n observations,

y_1, y_2, \dots, y_n , is the value of Φ that minimizes the sum of squares

$$Q_n(\Phi; y_1, y_2, \dots, y_n) = \sum_{t=p+1}^n (y_t - \phi_1 y_{t-1} - \dots - \phi_p y_{t-p})^2.$$

Equivalently, $\hat{\Phi}_{LS}^{(n)}$ is the solution of the linear system of equations

$$\partial Q_n / \partial \Phi = 2(X^T Y - X^T X \Phi) = 0 \quad (3.1)$$

where $\partial Q_n / \partial \Phi = (\partial Q_n / \partial \phi_1, \dots, \partial Q_n / \partial \phi_p)^T$ is the $p \times 1$ vector of partial derivatives, $Y = (y_{p+1}, \dots, y_n)^T$ and

$$X = \begin{pmatrix} y_p & y_{p-1} & \dots & y_1 \\ y_{p+1} & y_p & \dots & y_2 \\ \vdots & \vdots & & \vdots \\ y_{n-1} & y_{n-2} & \dots & y_{n-p} \end{pmatrix}.$$

If $X^T X$ is nonsingular, $\hat{\Phi}_{LS}^{(n)} = (X^T X)^{-1} X^T Y$ is the unique solution to (3.1).

In the case of i.i.d. innovations, $\hat{\Phi}_{LS}^{(n)}$ is consistent and asymptotically normal where the covariance matrix depends on only Φ , provided that the innovations have a finite variance (Whittle, 1953). If the autoregression is conditionally heteroscedastic this well-known result no longer holds. The least squares estimate is still consistent and asymptotically normal but the asymptotic covariance matrix depends

on the form of the conditional heteroscedasticity, as well as on Φ . The least squares estimate is not distribution-free.

The consistency and asymptotic normality of $\hat{\Phi}_{LS}^{(n)}$ can be derived using Klimko and Nelson's (1978) results on conditional least squares for general processes. For the CHAR processes under consideration, the assumptions of Klimko and Nelson (1978) reduce to the following:

A1: The CHAR process $\{y_t\}$ is a stationary and ergodic sequence of integrable random variables.

A2: $E(|\varepsilon_{p+1}y_{p+1-i}|)$ is finite for $1 \leq i \leq p$.

A3: The $p \times p$ covariance matrix V , defined by

$$V = \{E(y_{p+1-i}y_{p+1-j})\}, \quad 1 \leq i, j \leq p, \quad (3.2)$$

is positive definite.

A4: $E(\varepsilon_{p+1}^2 | y_{p+1-i}y_{p+1-j})$ is finite for $1 \leq i, j \leq p$.

The least squares estimate is consistent under assumptions A1, A2 and A3, while the proof of asymptotic normality requires the additional assumption A4.

Theorem 3.1 (Consistency of $\hat{\Phi}_{LS}^{(n)}$) - If A1, A2 and A3 hold for a CHAR process, then $\hat{\Phi}_{LS}^{(n)} \rightarrow \Phi^0$ (the true value of Φ) a.e., as $n \rightarrow \infty$.

Proof: The theorem follows as a direct application of

Theorem 3.1 of Klimko and Nelson (1978) and the uniqueness of the solution $\hat{\Phi}_{LS}^{(n)}$ a.e. as $n \rightarrow \infty$.

Theorem 3.2 (Asymptotic normality of $\hat{\Phi}_{LS}^{(n)}$) - Assume that A1, A2, A3 and A4 hold for the CHAR process under consideration. Then, as $n \rightarrow \infty$,

$$n^{\frac{1}{2}}(\hat{\Phi}_{LS}^{(n)} - \Phi^0) \rightarrow \text{MVN}_p(0, V^{-1}WV^{-1}) \quad (3.3)$$

where W is the $p \times p$ matrix

$$W = \{E(\varepsilon_{p+1}^2 y_{p+1-i} y_{p+1-j})\} \quad 1 \leq i, j \leq p. \quad (3.4)$$

Proof: The Taylor series expansion proof given by Klimko and Nelson (1978), for a general process, takes a particularly simple form here because $\partial Q_n / \partial \Phi$ can be written as

$$\partial Q_n / \partial \Phi = -2 \left(\sum_{t=p+1}^n \varepsilon_t y_{t-1}, \dots, \sum_{t=p+1}^n \varepsilon_t y_{t-p} \right)^T + (\Phi - \Phi^0)^T V_n$$

where V_n is the $p \times p$ matrix of second order derivatives:

$$V_n = 2 \left\{ \sum_{t=p+1}^n y_{t-i} y_{t-j} \right\} \quad 1 \leq i, j \leq p. \quad (3.5)$$

An application of the ergodic theorem gives $\frac{1}{2}n^{-1}V_n \rightarrow V$ a.e., as $n \rightarrow \infty$, while Billingsley's (1961) central limit theorem for martingales can be used to show that the first term in (3.5) is asymptotically normal, which proves the theorem.

It should be noted that, in general, the expression for the asymptotic covariance matrix in (3.3) depends on the form of the conditional variance. In particular, the asymptotic covariance matrix depends on the parameters $\alpha_1, \dots, \alpha_k$ in the CHARI model or β_1 in the CHARII model.

The verification of assumptions A1, A2, A3 and A4 is straightforward for the CHARI and CHARII processes. Assumption A1 holds under the hypotheses of Theorems 2.1 and 2.2, respectively. By Chebychev's inequality, A2 holds under the same conditions because they also imply that σ_y^2 is finite. Assumption A3 holds whenever A1 holds, provided σ_y^2 is positive. Finally, the existence of finite fourth moments for ε_t and y_t is sufficient for A4. This follows from Chebychev's inequality.

The significance of (3.3) is appreciated more readily in a specific application where the asymptotic covariance matrix can be evaluated explicitly. In the next section, the lowest order CHARI and CHARII models are used for illustration. The variance obtained using (3.3) is compared to the variance for i.i.d. innovations and a Monte Carlo simulation is used to investigate the small sample properties of the least squares estimate.

4. Examples

The simplest CHARI and CHARII processes are:

$$\text{CHARI}(1,1): y_t = \phi y_{t-1} + \varepsilon_t \quad \text{where } \varepsilon_t | \mathcal{Y}_{t-1} \sim N(0, \alpha_0 + \alpha_1 \varepsilon_{t-1}^2)$$

and

$$\text{CHARII}(1): y_t = \phi y_{t-1} + \varepsilon_t \quad \text{where } \varepsilon_t | \mathcal{Y}_{t-1} \sim N(0, \beta_0 + \beta_1 \phi^2 y_{t-1}^2).$$

If y_1, y_2, \dots, y_n is a sample from either process then a least squares estimate of ϕ is given by

$$\hat{\phi}_{LS}^{(n)} = \frac{\sum_{t=2}^n y_t y_{t-1}}{\sum_{t=2}^n y_{t-1}^2}.$$

The consistency and asymptotic normality of $\hat{\phi}_{LS}^{(n)}$ follow from Theorems 3.1 and 3.2, provided that the assumptions A1-A4 hold.

In the CHARI(1,1) process it is assumed that $|\phi| < 1$ and, $\alpha_0 > 0$ and $0 \leq \alpha_1 < 1$. Then, according to Theorem 2.1, the process is asymptotically stationary and ergodic and Theorem 3.1 can be applied to show that $\hat{\phi}_{LS}^{(n)}$ is consistent. The asymptotic normality follows if $E(\varepsilon_t^4)$ is finite, which implies that $E(y_t^4)$ is finite. It is easy to show that $E(\varepsilon_t^4)$ is finite if $\alpha_1 < 3^{\frac{1}{2}}$ (Engle, 1982). The matrices V and W , in (3.3), reduce to scalars where

$$V = \sigma_y^2 = \alpha_0 / \{(1 - \alpha_1)(1 - \phi^2)\},$$

from (2.9), and it is simple to show that

$$W = E(\varepsilon_t^2 y_{t-1}^2) = \alpha_0^2 / \{(1 - \alpha_1)(1 - \phi^2)\} + 2\alpha_0^2 \alpha_1 / \{(1 - \alpha_1)^2 (1 - 3\alpha_1^2)(1 - \alpha_1 \phi^2)\}$$

by evaluating a conditional expectation first. Substituting into

(3.3) gives

$$n^{\frac{1}{2}}(\hat{\phi}_{LS}^{(n)} - \phi) \rightarrow N[0, 1 - \phi^2 + 2\alpha_1(1 - \phi^2)^2 / \{(1 - \alpha_1\phi^2)(1 - 3\alpha_1^2)\}]. \quad (4.1)$$

Similar arguments can be provided for the CHARII(1) process. Accordingly, $\hat{\phi}_{LS}^{(n)}$ is consistent for a CHARII(1) process if $|\phi| < 1$, $\beta_0 > 0$ and $(1 + \beta_1)\phi^2 < 1$, so that by Theorem 2.2 the process is stationary and ergodic. If, in addition, $1 - \phi^4(1 + 6\beta_1 + 3\beta_1^2) > 0$ then y_t has a finite fourth moment. Substituting

$$V = \sigma_y^2 = \beta_0 / \{1 - (1 + \beta_1)\phi^2\},$$

from (2.11), and

$$\begin{aligned} W = E(\epsilon_t^2 y_{t-1}^2) &= \beta_0^2 / \{1 - (1 + \beta_1)\phi^2\} \\ &+ 3\beta_0^2 \beta_1 \phi^2 (1 + \phi^2 + \beta_1 \phi^2) / [\{1 - \phi^4(1 + 6\beta_1 + 3\beta_1^2)\} \{1 - (1 + \beta_1)\phi^2\}], \end{aligned}$$

which is calculated by conditioning on ψ_{t-1} first, into (3.3) gives

$$n^{\frac{1}{2}}(\hat{\phi}_{LS}^{(n)} - \phi) \rightarrow N[0, 1 - \phi^2 + 2\beta_1 \phi^2 (1 - \phi^4) / \{1 - \phi^4(1 + 6\beta_1 + 3\beta_1^2)\}]. \quad (4.2)$$

The first term in the variance of the asymptotic distributions, (4.1) and (4.2), corresponds to the variance for the usual autoregression with finite variance, i.i.d. innovations. The second term measures the increase in variance due to conditional heteroscedasticity. This increase can be considerable. However, since it is independent of α_0 or β_0 , the asymptotic variance is unaffected by increases in the unconditional variance of y_t that result from increasing α_0 or β_0 .

4.1 Monte Carlo Results

A Monte Carlo investigation of the small sample properties of the least squares estimate, $\hat{\phi}_{LS}^{(n)}$, was carried out for the CHARI(1,1) and CHARII(1) processes. Artificial sample values were generated recursively according to the formulas

$$\text{CHARI}(1,1): y_t = \phi y_{t-1} + \{\alpha_0 + \alpha_1 (y_{t-1} - \phi y_{t-2})^2\}^{\frac{1}{2}} z_t \quad (4.3)$$

$$\text{CHARII}(1): y_t = \phi y_{t-1} + \{\beta_0 + \beta_1 \phi^2 y_{t-1}^2\}^{\frac{1}{2}} z_t \quad (4.4)$$

where the z_t are i.i.d. $N(0,1)$ random variables, obtained by the method of Kinderman and Ramage (1976). The stationary initial distribution could not be found in closed form for either the CHARI(1,1) or CHARII(1) process. Instead of using the stationary initial distribution to generate the starting values, the final values in an initial block of 50, when $|\phi| \leq .5$, and 140, when $|\phi| > .5$, were used for y_0 and y_1 in (4.3) and y_0 in (4.4). A total of 1000 samples were generated for each parameter and sample size combination. The mean and variance of $\hat{\phi}_{LS}^{(n)}$, based on these replications, are given in Tables 1 and 2. The asymptotic values, calculated using (4.1) and (4.2), and the corresponding values for the classical autoregression are also given.

The Monte Carlo results show a bias typical (of the order $O(n^{-1})$) of least squares estimation. In cases where the fourth moment is large, convergence of the variance to its asymptotic value is slow. A comparison of the asymptotic variance, given by (4.1) or (4.2), and the corresponding value for i.i.d. innovations, $1 - \phi^2$, gives a measure of the effect of conditional heteroscedasticity. This can be large. For example, for the CHARI(1,1) process

Table 1: Mean and Variance of $\hat{\phi}_{LS}^{(n)}$ for CHARI(1,1)

** (based on 1000 replications)

$\phi=.5$			$\phi=.5$		
$\alpha_0=1$		$\alpha_1=.5$	$\alpha_0=1$		$\alpha_1=.2$
n	mean	n=variance	n	mean	n=variance
50	.4684	1.2539	50	.4772	.9633
100	.4798	1.5292	100	.4858	1.0431
150	.4874	1.5748	150	.4910	1.0044
500	.4945	2.1591	500	.4977	.9853
∞	.5000	3.3214	∞	.5000	1.0191
*	.5000	.7500	*	.5000	.7500

$\phi=.9$			$\phi=.9$		
$\alpha_0=1$		$\alpha_1=.5$	$\alpha_0=1$		$\alpha_1=.2$
n	mean	n=variance	n	mean	n=variance
50	.8624	.4205	50	.8660	.3005
100	.8740	.3361	100	.8814	.3019
150	.8863	.3655	150	.8890	.2315
500	.8943	.3324	500	.8963	.2298
∞	.9000	.4327	∞	.9000	.2096
*	.9000	.1900	*	.9000	.1900

* The last line corresponds to $\alpha_1=0$.

** The approximate standard error of the mean is less than 1% of the mean, in all cases, and the approximate standard error of n=variance is 4.5% for all cases. Approximate standard errors were calculated using the normal approximation (4.1).

Table 2: Mean and Variance of $\hat{\phi}_{LS}^{(n)}$ for CHARII(1)

** (based on 1000 replications)

$\phi=.1$			$\phi=.8$		
$\beta_0=1$		$\beta_1=20$	$\beta_0=1$		$\beta_1=.15625$
n	mean	n*variance	n	mean	n*variance
50	.0969	1.2400	50	.7535	.5757
100	.0940	1.2972	100	.7724	.5714
150	.0974	1.4006	150	.7809	.5645
500	.0972	1.4810	500	.7915	.6789
∞	.1000	1.4508	∞	.8000	1.0294
*	.1000	.9900	*	.8000	.3600

$\phi=.1$			$\phi=.4$		
$\beta_0=1$		$\beta_1=40$	$\beta_0=1$		$\beta_1=1.25$
n	mean	n*variance	n	mean	n*variance
50	.0912	1.6830	50	.3784	1.1606
100	.0969	1.9230	100	.3902	1.2285
150	.0989	2.0021	150	.3857	1.2878
500	.0973	2.1530	500	.3961	1.3210
∞	.1000	2.6031	∞	.4000	1.4284
*	.1000	.9900	*	.4000	.8400

* The last line corresponds to $\beta_1=0$.

** The maximum standard error in the mean is approximately 6.4%, 1.3% and .5% of the mean for $\phi=.1, .4$ and $.8$, respectively. The standard error of n*variance is approximately 4.5% for all cases.

with $\phi=.5$ and $\alpha_1=.5$ the variance is quadruple, and for the CHARII(1) process with $\phi=.1$ and $\beta_1=40$ the variance is double the value for i.i.d. innovations.

It is evident that failure to recognize conditional heteroscedasticity can lead to serious underestimation of the variance of the least squares estimate. Furthermore, the least squares estimate is inefficient when the process is actually a CHAR process. However, more efficient estimates, such as the maximum likelihood estimate, require a priori knowledge of the existence and form of the conditional heteroscedasticity. Initially, at least, inefficiency is unavoidable and the least squares estimate may be satisfactory. The identification and estimation of a CHAR process will be discussed in a future paper.

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References

- Billingsley, P. (1961). The Lindeberg-Levy theorem for martingales. Proc. Amer. Math. Soc., 12, 788-792.

- Carroll, R.J. and Ruppert, D. (1982). Robust estimation in heteroscedastic linear models. *Ann. Statist.*, 10, 429-440.
- Denby, L. and Martin, R.D. (1979). Robust estimation of the first-order autoregressive parameter. *J. Amer. Statist. Ass.*, 74, 140-146.
- Engle, R.F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrica*, 50, 987-1007.
- Fox, A.J. (1972). Outliers in time series. *J. R. Statist. Soc. B*, 34, 350-363.
- Kinderman, A.J. and Ramage, J.G. (1976). Computer generation of normal random variables. *J. Amer. Statist. Ass.*, 71, 893-896.
- Klimko, L.A. and Nelson, P.I. (1978). On conditional least squares estimation for stochastic processes. *Ann. Statist.*, 6, 629-642.
- Martin, R.D. (1982). The Cramer-Rao bound and robust M-estimates for autoregressions. *Biometrika*, 69, 437-442.
- Nemec, A.F.L. (1984). Conditionally heteroscedastic autoregressions. submitted to *Stochastic Processes*.
- Whittle, P. (1953). Estimation and information theory in stationary time series. *Arch. Math.*, 2, 423-434.